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Vector equilibrium problems with elastic demands and capacity constraints

S. J. Li · K. L. Teo · X. Q. Yang

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Abstract In this paper, a (weak) vector equilibrium principle for vector network problems with capacity constraints and elastic demands is introduced. A sufficient condition for a (weak) vector equilibrium flow to be a solution for a system of (weak) vector quasi-variational inequalities is obtained. By virtue of Gerstewitz's nonconvex separation functional ξ, a (weak) ξ-equilibrium flow is introduced. Relations between a weak vector equilibrium flow and a (weak) ξ-equilibrium flow is investigated. Relations between weak vector equilibrium flows and two classes of variational inequalities are also studied.

Keywords Vector traffic network equilibrium model · (weak) Vector equilibrium · (weak) ξ-equilibrium · Variational inequalities

1 Introduction

The earliest network equilibrium model was proposed by Wardrop for a transportation network. Since then, many other equilibrium models have been proposed in the economics literature (see [4,8,13]). In [5,6], Daniele et al. investigated a traffic equilibrium problem with capacity constraints in dynamic case and obtained sufficient and

S. J. Li (✉)
College of Mathematics and Science, Chongqing University, Chongqing 400044, China
e-mail: lisj@cqu.edu.cn

K. L. Teo
Department of Mathematics and Statistics, Curtin University of Technology, GPO Box U1987, Perth, W. A. 6845, Australia
e-mail: K.L.Teo@curtin.edu.au

X. Q. Yang
Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong
e-mail: mayangxq@polyu.edu.hk

necessary conditions for an equilibrium flow. In [14,15], Nagurney and Dong discussed a multiclass, multicriteria traffic equilibrium problem without capacity constraints and got equivalent relations between equilibrium flows and variational inequalities. But the criteria adopted in their studies were only the weighted sum of all criteria. As such, it is still a single cost function.

Recently, equilibrium models based multicriteria consideration or vector-valued cost functions have been proposed (see [1,2,9,10,16]). In [1], Chen and Yen first introduced a vector equilibrium principle for a vector traffic network without capacity constraints, which is a generalization of the classical Wardrop's user-optimizing principle. In [16], Yang and Goh investigated equivalent relations among a (weak) vector equilibrium principle, a class of (weak) vector variational inequalities and a class of vector optimization problems for a vector traffic network without capacity constraints. In [2], Chen et al. introduced an ξ_{ea}-functional and an ξ_{ea}-equilibrium principle for a vector traffic network without capacity constraints. They proved an equivalent relation between a weak vector equilibrium flow and an ξ_{ea}-equilibrium flow and obtained a sufficient and necessary condition for a weak vector equilibrium flow to be a solution of a class of variational inequalities. In [11], Li et al. introduced a generalized vector equilibrium principle for a vector traffic network with capacity constraints and obtained a necessary and sufficient condition for the generalized vector flow to be a minimum vector cost flow.

When the cost function is a scalar function, it is known that finding an equilibrium flow is equivalent to solving a class of variational inequalities. However, when the cost function is vector-valued, finding a weak vector equilibrium flow is, in general, not necessarily equivalent to solving a weak vector variational inequality problem. Naturally, we hope to obtain a sufficient and necessary condition for a weak vector equilibrium flow. An ξ_{ea}-equilibrium principle for the vector network equilibrium problem with capacity constraints and elastic demands can be introduced as in [2] (see Definition 3.2). Unfortunately, we can only establish that an ξ_{ea}-equilibrium flow is a weak vector equilibrium one, but not vice versa. Therefore, in this paper, we also introduce a weak ξ-equilibrium principle, which is weaker than the ξ_{ea}-equilibrium principle introduced in [2] in the sense that the element a in ξ_{ea}-equilibrium is not fixed. As such, we are able to obtain a sufficient and necessary condition for a weak vector equilibrium flow to be a weak ξ-equilibrium flow. Furthermore, we introduce a class of variational inequalities by virtue of ξ-functional and establish an equivalent relation between a weak vector equilibrium flow and a solution of a scalar variational inequality problem.

The outline of the paper is as follows. In Sect. 2, a (weak) vector equilibrium principle is introduced. Then, sufficient conditions for a (weak) vector equilibrium flow to be a solution for the system of (weak) quasi-variational inequalities are obtained. In Sect. 3, the (weak) ξ-equilibrium principle is introduced. Relations between weak vector equilibrium flows and (weak) ξ-equilibrium flows are investigated. In Sect. 4, equivalent relations between (respectively, weak) ξ-equilibrium flows and (respectively, another) classes of variational inequalities are also discussed.

2 Vector equilibrium principle with capacity and elastic demand

Consider a supply-demand network $G = [\mathcal{N}, \mathcal{L}]$, where $\mathcal{N}$ denotes the set of nodes in the network and $\mathcal{L}$ is the set of directed arcs. Let a denote an arc of the network connecting a pair of nodes, and p denote a path, assumed to be acyclic, consisting of a sequence of arcs connecting an origin/destination (O/D) pair of nodes. The set of O/D pairs is denoted by W and the set of available paths joining the O/D pair w is denoted by P_w. Let

$$n = |\mathcal{L}|, \qquad P = \bigcup_{w \in W} P_w \quad \text{and} \quad m = |P|.$$

Assume that there are q classes of products to traverse in the network with a typical product class denoted by j. Let g_a^j and f_p^j denote the flows of product j on arc a and on path p, respectively. Group the flows on all arcs for product j into the n-dimensional column vector g^j with components: $\left\{g_a^j, \ldots, g_n^j\right\} \in \mathcal{R}^n$, which is called as an arc flow of product j on the network, and group the flows on all paths for product j into the m-dimensional column vector f^j with components: $\left\{f_p^j, \ldots, f_m^j\right\} \in \mathcal{R}^m$, which is called as a path flow of product j on the network. The relationship between the arc flows and the path flows for product j is

$$g_a^j = \sum_{p \in P} \delta_{ap} f_p^j,$$

where $\delta_{ap} = 1$, if arc a is contained in path p, and 0, otherwise. Let

$$g = ((g^1)^{\mathrm{T}}, \ldots, (g^q)^{\mathrm{T}})^{\mathrm{T}} \in \mathcal{R}^{qn} \qquad \text{and } f = \left((f^1)^{\mathrm{T}}, \ldots, (f^q)^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathcal{R}^{qm},$$

where the superscript T denotes transpose. f is called as a flow on the network.

Let $c_a^j(g)$: $\mathcal{R}^n \to \mathcal{R}^l$ be a vector-valued cost function for arc a and product j and $c_p^j(f)$: $\mathcal{R}^m \to \mathcal{R}^l$ be a vector-valued cost function for path p and product j, where

$$c_p^j(f) = \sum_{a \in \mathcal{L}} \delta_{ap} c_a^j(g).$$

Similarly, group vector cost functions on all paths for product j into an $l \times m$ matrix-valued function $c^j(f)$ with column components: $\left\{c_p^j(f), \ldots, c_m^j(f)\right\} \in S^{l \times m}$ and let

$$c(f) = (c^1(f), \ldots, c^q(f)) \in S^{l \times mq}, \tag{1}$$

where $S^{l \times r}$ denotes the set of real $l \times r$ matrices.

Suppose that the demand of network flow is not fixed for each O/D pair w and product j. In general, it depends on the costs for all O/D pair. Thus, by (1), we assume directly that the demand is a function of a flow f (see [12]). We say that a flow h satisfies demands for the flow f if

$$\sum_{p \in P_w} h_p^j = d_w^j(f), \quad \forall w \text{ and } j, \tag{2}$$

where $d_w^j(f) \in \mathcal{R}_+$ is a given demand for O/D pair w, product j and the flow f, that is, the travel demand of product j for an O/D pair and the flow f is equal to the sum of the flows of product j on paths connecting the O/D pair. Note that f is a given flow. It may not satisfy demands for oneself, i.e., there may be j_0 and w_0 such that

$$\sum_{p \in P_{w_0}} f_p^{j_0} \neq d_{w_0}^{j_0}(f).$$

Let λ_p^j and μ_p^j be lower and upper capacity constraints of path p for product j, respectively, namely,

$$\lambda_p^j \leq f_p^j \leq \mu_p^j.$$

We group lower and upper capacity constraints of all paths for product j into two m-dimensional column vectors λ^j and μ^j with components: $\{\lambda_p^j, \ldots, \lambda_m^j\}$ and $\{\mu_p^j, \ldots, \mu_m^j\}$ $\in \mathcal{R}^m$, respectively. Let

$$\lambda = ((\lambda^1)^{\mathrm{T}}, \ldots, (\lambda^q)^{\mathrm{T}})^{\mathrm{T}} \qquad \text{and} \qquad \mu = ((\mu^1)^{\mathrm{T}}, \ldots, (\mu^q)^{\mathrm{T}})^{\mathrm{T}}.$$

For the l-dimensional Euclidean space $\mathcal{R}^l$, we denote the orderings induced by $\mathcal{R}_+^l$ as follows:

$$\begin{aligned} x \preceq y \quad & \text{iff } \ y - x \in \mathcal{R}_+^l, \\ x \prec y \quad & \text{iff } \ y - x \in \mathrm{int}\mathcal{R}_+^l, \end{aligned}$$

where $\mathrm{int}\mathcal{R}_+^l$ is the interior of $\mathcal{R}_+^l$.

A flow h satisfying the demand requirements (2) and capacity constraints is called a feasible flow for the flow f, namely,

$$\lambda \preceq h \preceq \mu$$

and for every $w \in W$ and product j,

$$\sum_{p \in P_w} h_p^j = d_w^j(f).$$

The set of feasible flows is given by

$$\mathcal{K}(f) = \left\{ h \mid \lambda \preceq h \preceq \mu, \text{ and } \sum_{p \in P_w} h_p^j = d_w^j(f), \text{ for every } w \in W, j = 1, \ldots, q \right\}$$

and it is called the feasible set for the flow f. $\mathcal{K}(f)$ is clearly a closed convex set for every fixed f. Now we introduce two vector equilibrium principles for the vector traffic network equilibrium model with capacity constraints and elastic demands.

Definition 2.1 (Vector equilibrium principle) A flow $f \in \mathcal{K}(f)$ is said to be in vector equilibrium if for any O/D pair w, we have

$$\forall p, p' \in P_w, \quad c_{p'}^j(f) - c_p^j(f) \in \mathcal{R}_+^l \setminus \{0\} \Longrightarrow f_p^j = \mu_p^j \quad \text{or} \quad f_{p'}^j = \lambda_{p'}^j, \ j = 1, \ldots, q. \tag{3}$$

Definition 2.2 (Weak vector equilibrium principle) A flow $f \in \mathcal{K}(f)$ is said to be in weak vector equilibrium if for any O/D pair w, we have

$$\forall p, p' \in P_w, \quad c_{p'}^j(f) - c_p^j(f) \in \mathrm{int}\mathcal{R}_+^l \Longrightarrow f_p^j = \mu_p^j \quad \text{or} \quad f_{p'}^j = \lambda_{p'}^j, \ j = 1, \ldots, q. \tag{4}$$

Remark 2.1 If $q = 1$, the demand is fixed and the cost function is a scalar function, the vector equilibrium flows in Definitions 2.1 and 2.2 reduce assumption condition (3) in [5], which is equivalent to an equilibrium flow defined in [5]. Thus, Definitions 2.1 and 2.2 are generalizations of equilibrium flows defined in [5].

To investigate sufficient conditions for a flow to be in weak vector equilibrium, we introduce the following systems of vector quasi-variational inequalities:

$$\text{(SVQVI)} \qquad \begin{cases} \text{Find } f \in \mathcal{K}(f) \text{ such that for } j = 1, \ldots, q, f^j \in K^j(f) \quad \text{and} \\ \langle c^j(f), h^j - f^j \rangle \notin -\mathcal{R}^l \backslash \{0\}, \text{ for all } h^j \in \mathcal{K}^j(f), \end{cases}$$

where

$$\mathcal{K}^j(f) = \left\{ h^j \mid \lambda^j \preceq h^j \preceq \mu^j, \text{ and } \sum_{p \in P_w} h^j_p = d^j_w(f), \text{ for every } w \in W \right\}$$

and systems of weak vector quasi-variational inequalities:

$$\text{(SVQVI)}_w \qquad \begin{cases} \text{Find } f \in \mathcal{K}(f) \text{ such that for } j = 1, \ldots, q, f^j \in K^j(f) \text{ and} \\ \langle c^j(f), h^j - f^j \rangle \notin -\operatorname{int}\mathcal{R}^l_+ \text{ for all } h^j \in \mathcal{K}^j(f). \end{cases}$$

Theorem 2.1 *If a flow $f \in \mathcal{K}(f)$ solves the problem (*SVQVI*), then the flow f is a vector equilibrium flow.*

Proof Suppose that f is not a vector equilibrium flow. Then, there exist $1 \le j \le q, w \in W$ and $t, s \in P_w$ such that

$$c^j_s(f) - c^j_t(f) \in \mathcal{R}^l_+ \backslash \{0\}, \quad f^j_t < \mu^j_t \quad \text{and} \quad f^j_s > \lambda^j_s. \tag{5}$$

Construct a flow h as follows:

$$h^i_r = \begin{cases} f^i_r, & i \neq j, \\ f^j_r, & r \neq t, s \text{ and } i = j, \\ f^j_t + \varepsilon, & r = t \text{ and } i = j, \\ f^j_s - \varepsilon, & r = s \text{ and } i = j, \end{cases}$$

where

$$0 < \varepsilon \le \min \left\{ \mu^j_t - f^j_t, f^j_s - \lambda^j_s \right\}.$$

Then, we have

$$h \in \mathcal{K}(f).$$

It follows readily that

$$\begin{aligned} \sum_{p \in P} \left\langle c^j_p(f), h^j_p - f^j_p \right\rangle &= \left\langle c^j_t(f), f^j_t + \varepsilon \right\rangle - \left\langle c^j_t(f), f^j_t \right\rangle + \left\langle c^j_s(f), f_s - \varepsilon \right\rangle - \left\langle c^j_s(f), f_s \right\rangle \\ &= \left\langle c^j_t(f), \varepsilon \right\rangle - \left\langle c^j_s(f), \varepsilon \right\rangle \\ &= \left\langle c^j_t(f) - c^j_s(f), \varepsilon \right\rangle \in -\mathcal{R}^l_+ \backslash \{0\}, \end{aligned}$$

which contradicts the assumption condition that the flow f solves the problem (SVQVI). □

Following the proof of Theorem 2.1, we can establish a similar sufficient condition for a weak vector equilibrium flow.

Theorem 2.2 *If a flow $f \in \mathcal{K}(f)$ solves $(SVQVI)_w$, then the flow f is a weak vector equilibrium flow.*

3 ξ-equilibrium principles

It follows from Theorems 2.1 and 2.2 that the solutions of the problem (SVQVI) (respectively, $(\text{SVQVI})_w$) must be vector equilibrium flows (respectively, weak vector equilibrium flows). However, the converse relation does not necessarily hold. In this section, we shall use Gerstewitz's nonconvex separation functional to introduce a (weak) ξ-equilibrium flow. We also discuss the equivalent relation between weak vector equilibrium flows and (weak) ξ-equilibrium flows. Now we present a nonconvex separation functional and a weakly efficient point.

Given a fixed $e \in \text{int}\mathcal{R}^l$ and $\alpha \in \mathcal{R}^l$, the Gerstewitz's nonconvex separation functional $\xi_{ea}: \mathcal{R}^l \to \mathcal{R}$ is defined by:

$$\xi_{e\alpha}(y) = \min\{\lambda \in \mathcal{R}: y \in \alpha + \lambda e - \mathcal{R}^l_+\}, \quad \forall y \in \mathcal{R}^l.$$

It follows from the proof process of Theorem 2.1 in [7] that $\xi_{e\alpha}$ is continuous on $\mathcal{R}^l$ and strictly monotone on $\text{int}\mathcal{R}^l_+$ for any fixed $\alpha \in \mathcal{R}^l$.

Let A be a nonempty subset in $\mathcal{R}^l$. A point $y \in A$ is said to be a weakly efficient point of A if

$$A \bigcap (y - \text{int}\mathcal{R}^l_+) = \emptyset.$$

By $w-$ Eff(A) we denote the set of all weakly efficient points of A.

From Corollary 3.1 and Remark 3.1 in [7], one has the following result.

Lemma 3.1 *Let $e \in \text{int}\mathcal{R}^l_+$. Then, $y^0 \in w - \text{Eff}(A)$ if and only if $y^0 \in A$ and the functional, given by*

$$\xi_{ey^0}(y) = \min\{\eta \in \mathcal{R} \mid y \in y^0 + \eta e - \mathcal{R}^l_+\},$$

satisfies the following conditions:

$$\xi_{ey^0}\left(y^0\right) = 0, \;\; \xi_{ey^0}(A) \geq 0 \quad \textit{and} \quad \xi_{ey^0}\left(y^0 - \text{int}\mathcal{R}^l_+\right) < 0,$$

where $\xi_{ey^0}(F) \geq 0$ denotes $\xi_{ey^0}(y) \geq 0, \forall y \in F$.

We introduce now two kinds of ξ_{ea}-equilibrium flows. Using the two concepts we can get some sufficient and necessary conditions of weak vector equilibrium flows.

Definition 3.1 Let $e \in \text{int}\mathcal{R}^l_+$. A flow $f \in \mathcal{K}(f)$ is said to be in weak ξ-equilibrium if for any O/D pair w, we have

$$\forall p, p' \in P_w, \quad \xi_{ec^j_{p'}(f)}(c^j_p(f)) < 0 \Longrightarrow f^j_p = \mu^j_p \quad \text{or} \quad f^j_{p'} = \lambda^j_{p'}, \; j = 1, \ldots, q. \tag{6}$$

Definition 3.2 Let $e \in \text{int}\mathcal{R}^l_+$. A flow $f \in \mathcal{K}(f)$ is said to be in ξ-equilibrium if there exists an $\alpha \in \mathcal{R}^l$ such that, for any O/D pair w, we have

$$\forall p, p' \in P_w, \quad \xi_{e\alpha}(c^j_p(f)) < \xi_{e\alpha}(c^j_{p'}(f)) \Longrightarrow f^j_p = \mu^j_p \quad \text{or} \quad f^j_{p'} = \lambda^j_{p'}, \; j = 1, \ldots, q. \tag{7}$$

Remark 3.1 In [3], Cheng and Wu discussed a multiclass, multicriteria supply-demand network equilibrium model without capacity constraints and obtained a necessary and sufficient condition of the weak vector equilibrium principle. However, for a multiclass, vector supply-demand network equilibrium model with capacity constraints, if we introduce ξ_{ea}-equilibrium principle similar to the idea of [3] or Definition 3.2, we can only obtain a sufficient condition. Thus, it is necessary to introduce a weak ξ-equilibrium principle in order to obtain a sufficient and necessary condition.

Theorem 3.1 *A flow $f \in \mathcal{K}(f)$ is in weak vector equilibrium if and only if $f \in \mathcal{K}(f)$ is in weak ξ-equilibrium.*

Proof Assume that $f \in \mathcal{K}(f)$ is in weak ξ-equilibrium, i.e., (6) holds, but f is not in weak vector equilibrium. Then, there exist an O/D pair w, paths $p, p' \in P_w$ and a product j such that

$$c^j_{p'}(f) - c^j_p(f) \in \operatorname{int}\mathcal{R}^l_+,$$

$$f^j_p < \mu^j_p \quad \text{and} \quad f^j_{p'} > \lambda^j_{p'}. \tag{8}$$

It follows from the strict $\operatorname{int}\mathcal{R}^l_+$-monotonicity of $\xi_{e\alpha}$ functional for any $\alpha \in \mathcal{R}^l$ that

$$\xi_{ec^j_{p'}(f)}(c^j_p(f)) < \xi_{ec^j_{p'}(f)}(c^j_{p'}(f)) = 0.$$

By (6), one has $f^j_p = \mu^j_p$ or $f^j_{p'} = \lambda^j_{p'}$, which contradicts (8).

Conversely, assume that $f \in \mathcal{K}(f)$ is in weak vector equilibrium, but f is not in weak ξ-equilibrium. Then, there exist an O/D pair w, paths $p, p' \in P_w$ and a product j such that

$$\xi_{ec^j_{p'}(f)}(c^j_p(f)) < 0, \tag{9}$$

$$f^j_p < \mu^j_p \quad \text{and} \quad f^j_{p'} > \lambda^j_{p'}. \tag{10}$$

If $c^j_p(f) \prec c^j_{p'}(f)$, i.e., $c^j_{p'}(f) - c^j_p(f) \in \operatorname{int}\mathcal{R}^l_+$, it follows from the weak vector equilibrium property of the flow f that

$$f^j_p = \mu^j_p \quad \text{or } f^j_{p'} = \lambda^j_{p'},$$

which contradicts (10).
If $c^j_{p'}(f) - c^j_p(f) \notin \operatorname{int}\mathcal{R}^l_+$, then

$$c^j_p(f) - c^j_{p'}(f) \notin -\operatorname{int}\mathcal{R}^l_+.$$

Set

$$A = \left\{ c^j_s(f) \mid c^j_s(f) \notin c^j_{p'}(f) - \operatorname{int}\mathcal{R}^l_+, s \in P_w \right\}.$$

Then, we have

$$c^j_{p'}(f),\ c^j_p(f) \in A \quad \text{and} \quad c^j_{p'}(f) \in w-\ \mathrm{Eff}(A).$$

From Lemma 3.1, one has

$$\xi_{ec^j_{p'}(f)}\left(c^j_p(f)\right) \geq 0,$$

which contradicts (9). Thus, f is a weak ξ-equilibrium flow and the proof is complete. □

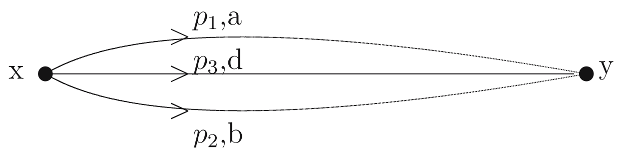

Fig. 1 Network topology for an example

Theorem 3.2 *If $f \in \mathcal{K}(f)$ is in ξ-equilibrium, then the flow f is in weak vector equilibrium.*

Proof Suppose that for the O/D pair w, paths $p, p' \in P_w$ and the product j, we have

$$c^j_{p'}(f) - c^j_p(f) \in \mathrm{int}\mathcal{R}^l_+.$$

It follows from the strict $\mathrm{int}\mathcal{R}^l_+$-monotonicity property of the $\xi_{e\alpha}$ functional that

$$\xi_{e\alpha}(c^j_p(f)) < \xi_{e\alpha}(c^j_{p'}(f)).$$

From Definition 3.2, one has

$$f^j_p = \mu^j_p \quad \text{or} \quad f^j_{p'} = \lambda^j_{p'}.$$

Thus, the flow $f \in \mathcal{K}(f)$ is in weak vector equilibrium. □

By Theorems 3.1 and 3.2, we get the following corollary.

Corollary 3.1 *If a flow $f \in \mathcal{K}(f)$ is in ξ-equilibrium, then the flow f is in weak ξ-equilibrium.*

Note that when $f \in \mathcal{K}(f)$ is in weak ξ-equilibrium, f may not be in ξ-equilibrium. The following example explains this situation.

Example 3.1 Consider the network problem depicted in Fig. 1, which consists two nodes x and y, three arcs a, b and d and a single O/D pair $w = (x, y)$.

Assume that there is only one product to traverse in the network and upper capacities of the three paths $\{p_1, p_2, p_3\}$ are 4, 3 and 3, respectively, and their lower capacities are all zero. The travel demand for w is fixed and $d_w = 6$. The arc cost functions from $\mathcal{R}^3$ to $\mathcal{R}^2$ are, respectively,

$$c_a(g) = \begin{pmatrix} 6g_a \\ 3g_a \end{pmatrix}, \quad c_b(g) = \begin{pmatrix} 5g_b \\ 4g_b \end{pmatrix} \quad \text{and} \quad c_d(g) = \begin{pmatrix} 7g_d \\ g_d \end{pmatrix},$$

and the path cost functions from $\mathcal{R}^3$ to $\mathcal{R}^2$ are, respectively

$$c_{p_1}(f) = \begin{pmatrix} 6f_{p_1} \\ 3f_{p_1} \end{pmatrix}, \quad c_{p_2}(f) = \begin{pmatrix} 5f_{p_2} \\ 4f_{p_2} \end{pmatrix} \quad \text{and} \quad c_{p_3}(f) = \begin{pmatrix} 7f_{p_3} \\ f_{p_3} \end{pmatrix}.$$

We have

$$\lambda = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}, \quad W = \{w = (x, y)\}, \quad P_w = P = \{p_1, p_2, p_3\}.$$

Take

$$f^*_{p_1} = 2, \qquad f^*_{p_2} = 2 \quad \text{and} \quad f^*_{p_3} = 2.$$

We have

$$c_{p_1}(f^*) = \begin{pmatrix} 12 \\ 6 \end{pmatrix}, \quad c_{p_2}(f^*) = \begin{pmatrix} 10 \\ 8 \end{pmatrix} \quad \text{and} \quad c_{p_3}(f^*) = \begin{pmatrix} 14 \\ 2 \end{pmatrix}.$$

Take $e = (1,1)^T \in \mathrm{int}\mathcal{R}^2_+$ and $\alpha = (\alpha_1, \alpha_2)^{\mathrm{T}} \in \mathcal{R}^2$. It follows from Chen et al [2] that the $\xi_{e\alpha}$-functional may be denoted as

$$\xi_{e\alpha}(y) = \max\{y_1 - \alpha_1, y_2 - \alpha_2\}.$$

Then,

$$\begin{aligned}
\xi_{ec_{p_1}(f^*)}(c_{p_2}(f^*)) &= \max\{10 - 12, 8 - 6\} = 2, \\
\xi_{ec_{p_1}(f^*)}(c_{p_3}(f^*)) &= \max\{14 - 12, 2 - 6\} = 2, \\
\xi_{ec_{p_2}(f^*)}(c_{p_1}(f^*)) &= \max\{12 - 10, 6 - 8\} = 2, \\
\xi_{ec_{p_2}(f^*)}(c_{p_3}(f^*)) &= \max\{14 - 10, 2 - 8\} = 4, \\
\xi_{ec_{p_3}(f^*)}(c_{p_1}(f^*)) &= \max\{12 - 14, 6 - 2\} = 4, \\
\xi_{ec_{p_3}(f^*)}(c_{p_2}(f^*)) &= \max\{10 - 14, 8 - 2\} = 6.
\end{aligned}$$

Thus, f^* is a weak ξ-equilibrium flow. On the other hand, for any $\alpha \in \mathcal{R}^2$, we have

$$\begin{aligned}
\xi_{e\alpha}(c_{p_1}(f^*)) &= \max\{12 - \alpha_1, 6 - \alpha_2\}, \\
\xi_{e\alpha}(c_{p_2}(f^*)) &= \max\{10 - \alpha_1, 8 - \alpha_2\}, \\
\xi_{e\alpha}(c_{p_3}(f^*)) &= \max\{14 - \alpha_1, 2 - \alpha_2\}.
\end{aligned}$$

If $\xi_{e\alpha}(c_{p_1}(f^*)) = 6 - \alpha_2$, then

$$\xi_{e\alpha}(c_{p_1}(f^*)) < \xi_{e\alpha}(c_{p_2}(f^*)). \tag{11}$$

We have

$$f^*_{p_1} = 2 < \mu_{p_1} = 4 \quad \text{and} \quad f^*_{p_2} = 2 > \lambda_{p_2} = 0. \tag{12}$$

If $\xi_{e\alpha}(c_{p_1}(f^*)) = 12 - \alpha_1$, then

$$\xi_{e\alpha}(c_{p_1}(f^*)) < \xi_{e\alpha}(c_{p_3}(f^*)). \tag{13}$$

We have

$$f^*_{p_1} = 2 < \mu_{p_1} = 4 \quad \text{and} \quad f^*_{p_3} = 2 > \lambda_{p_3} = 0. \tag{14}$$

Thus, it follows from (11)–(14) that f^* is not in ξ-equilibrium.

4 Equivalent relationships

In this section, we shall discuss the equivalent relation of weak vector equilibrium flows, i.e., weak ξ-equilibrium flows, and solutions for a class of variational inequalities. We shall also investigate the equivalent relation of ξ-equilibrium flows and another class of variational inequalities.

Theorem 4.1 *Let $e \in \mathrm{int}\mathcal{R}^l_+$. A flow $f \in \mathcal{K}(f)$ is in weak vector equilibrium if and only if $f \in \mathcal{K}(f)$ solves the following variational inequality:*

$$\sum_{j=1}^{q}\sum_{w\in W}\sum_{p\in P_w}\left\langle \min_{v\in A^j_w} \xi_{ec^j_v(f)}(c^j_p(f)), h^j_p - f^j_p \right\rangle \geq 0, \quad \forall h \in \mathcal{K}(f), \tag{15}$$

where

$$A^j_w := \left\{ v \in P_w \mid f^j_v > \lambda^j_v \right\}.$$

Proof Assume that $f \in \mathcal{K}(f)$ is not a weak vector equilibrium flow. Then, there exist $1 \leq j \leq q, w \in W$ and $t, s \in P_w$ such that

$$c^j_s(f) - c^j_t(f) \in \ \mathrm{int}\mathcal{R}^l_+, \quad f^j_t < \mu^j_t \quad \text{and} \quad f^j_s > \lambda^j_s. \tag{16}$$

Construct a flow h as follows:

$$h^i_r = \begin{cases} f^i_r, & i \neq j, \\ f^j_r, & r \neq t, s \text{ and } i = j, \\ f^j_t + \varepsilon, & r = t \text{ and } i = j, \\ f^j_s - \varepsilon, & r = s \text{ and } i = j. \end{cases}$$

where

$$0 < \varepsilon \leq \min\{\mu^j_t - f^j_t, f^j_s - \lambda^j_s\}.$$

Then, $h \in \mathcal{K}(f)$. It follows readily that

$$\begin{aligned} &\sum_{j=1}^{q}\sum_{w\in W}\sum_{p\in P_w}\left\langle \min_{v\in A^j_w} \xi_{ec^j_v(f)}(c^j_p(f)), h^j_p - f^j_p \right\rangle \\ &= \left\langle \min_{v\in A^j_w} \xi_{ec^j_v(f)}(c^j_t(f)), \varepsilon \right\rangle - \left\langle \min_{v\in A^j_w} \xi_{ec^j_v(f)}(c^j_s(f)), \varepsilon \right\rangle \\ &= \left\langle \min_{v\in A^j_w} \xi_{ec^j_v(f)}(c^j_t(f)) - \min_{v\in A^j_w} \xi_{ec^j_v(f)}(c^j_s(f)), \varepsilon \right\rangle. \end{aligned}$$

Since the $\xi_{e\alpha}$-functional is strict $\mathrm{int}\mathcal{R}^l_+$-monotone and A is a finite set, it follows from (16) that

$$\min_{v\in A^j_w} \xi_{ec^j_v(f)}\left(c^j_t(f)\right) - \min_{v\in A^j_w} \xi_{ec^j_v(f)}(c^j_s(f)) < 0.$$

Thus, we have

$$\sum_{j=1}^{q}\sum_{w\in W}\sum_{p\in P_w}\left\langle \min_{v\in A^j_w} \xi_{ec^j_c(f)}(c^j_p(f)), h^j_p - f^j_p \right\rangle < 0,$$

which contradicts (15). Thus, f is a weak vector equilibrium flow.

Conversely, for any product j and $w \in W$, set

$$B^j_w := \left\{ u \in P_w \mid f^j_u < \mu^j_u \right\}.$$

Since f is in weak vector equilibrium,

$$c_u^j(f) - c_v^j(f) \notin -\mathrm{int}\mathcal{R}_+^l \quad \text{for all } u \in B_w^j, \quad v \in A_w^j.$$

It follows from the proof process of Theorem 3.1 that

$$\xi_{ec_v^j(f)}(c_u^j(f)) \geq 0 \quad \text{for all } u \in B_w^j, \quad v \in A_w^j.$$

So, there exists a $\gamma_w^j \geq 0$ such that

$$\min_{u \in B_w^j} \min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_u^j(f)) = \gamma_w^j.$$

Let $h \in \mathcal{K}(f)$ be arbitrary. Then, for every $r \in P_w$, and $1 \leq j \leq q$, we consider three cases:

Case 1 If $\min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_r^j(f)) < \gamma_w^j$, then $r \notin B_w^j$. Hence, $f_r^j = \mu_r^j,\ \ h_r^j - f_r^j \leq 0$ and

$$\left(\min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_r^j(f)\right) - \gamma_w^j)(h_r^j - f_r^j) \geq 0. \tag{17}$$

Case 2 $\min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_r^j(f)) > \gamma_w^j$, then we have

$$\min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_r^j(f)) > 0. \tag{18}$$

Since $\min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_p^j(f)) \leq 0,\ \forall p \in A_w^j$, it follows from (18) that $r \notin A_w^j$. Hence, $f_r^j = \lambda_r^j,\ \ h_r^j - f_r^j \geq 0$ and

$$\left(\min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_r^j(f)\right) - \gamma_w^j)(h_r^j - f_r^j) \geq 0. \tag{19}$$

Case 3 If $\min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_r^j(f)) = \gamma_w^j$, then we have

$$\left(\min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_r^j(f)\right) - \gamma_w^j)(h_r^j - f_r^j) = 0. \tag{20}$$

From (17), (19) and (20),

$$\sum_{j=1}^{q} \sum_{w \in W} \sum_{p \in P_w} \left\langle \min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_p^j(f)), h_p^j - f_p^j \right\rangle \geq \sum_{j=1}^{q} \sum_{w \in W} \gamma_w^j (d_w^j - d_w^j) = 0.$$

Thus, the formula (15) holds. □

From Theorems 3.1 and 4.1, the following corollary holds.

Corollary 4.1 *Let $e \in int\mathcal{R}_+^l$. A flow $f \in \mathcal{K}(f)$ is a weak ξ-equilibrium flow if and only if $f \in \mathcal{K}(f)$ solves the following variational inequality:*

$$\sum_{j=1}^{q} \sum_{w \in W} \sum_{p \in P_w} \left\langle \min_{v \in A_w^j} \xi_{ec_v^j(f)}(c_p^j(f)), h_p^j - f_p^j \right\rangle \geq 0, \quad \forall h \in \mathcal{K}(f),$$

Theorem 4.2 *Let $e \in int\mathcal{R}^l_+$. A flow $f \in \mathcal{K}(f)$ is in ξ-equilibrium if and only if there is an $a \in \mathcal{R}^l$ such that the flow $f \in \mathcal{K}(f)$ solves the following variational inequality:*

$$\sum_{j=1}^{q}\sum_{w\in W}\sum_{p\in P_w}\left\langle \xi_{ea}(c_p^j(f)), h_p^j - f_p^j \right\rangle \geq 0, \quad \forall h \in \mathcal{K}(f). \tag{21}$$

Proof Similar to the proof of Theorem 4.1, suppose that f is not a ξ-equilibrium flow. Then, for any $a \in \mathcal{R}^l$, there exist $1 \leq j \leq q, w \in W$ and $t, s \in P_w$ such that

$$\xi_{ea}(c_t^j(f)) < \xi_{ea}(c_s^j(f)), \quad f_t^j < \mu_t^j \quad \text{and} \quad f_s^j > \lambda_s^j.$$

Construct a flow h as follows:

$$h_r^i = \begin{cases} f_r^i, & i \neq j, \\ f_r^j, & r \neq t, s \text{ and } i = j, \\ f_t^j + \varepsilon, & r = t \text{ and } i = j, \\ f_s^j - \varepsilon, & r = s \text{ and } i = j, \end{cases}$$

where

$$0 < \varepsilon \leq \min\{\mu_t^j - f_t^j, f_s^j - \lambda_s^j\}.$$

Then, $h \in \mathcal{K}(f)$ and

$$\sum_{j=1}^{q}\sum_{w\in W}\sum_{p\in P_w}\left\langle \xi_{ea}(c_p^j(f)), h_p^j - f_p^j \right\rangle = \left\langle \xi_{ea}(c_t^j(f)) - \xi_{ea}(c_s^j(f)), \varepsilon \right\rangle < 0,$$

which contradicts (21). Thus, $f \in \mathcal{K}(f)$ is a ξ-equilibrium flow.

Conversely, for any product j and $w \in W$, set

$$A_w^j := \left\{ v \in P_w \mid f_v^j > \lambda_v^j \right\} \quad \text{and} \quad B_w^j := \left\{ u \in P_w \mid f_u^j < \mu_u^j \right\}.$$

It follows from the definition of the ξ-equilibrium flow that

$$\xi_{ea}\left(c_u^j(f)\right) \geq \xi_{ea}\left(c_v^j(f)\right), \quad \forall u \in B_w^j, v \in A_w^j.$$

So, there exists a $\gamma_w^j \in \mathcal{R}$ such that

$$\min_{u\in B_w^j} \xi_{ea}(c_u^j(f) \geq \gamma_w^j \geq \max_{v\in A_w^j} \xi_{ea}(c_v^j(f)).$$

Similar to the proof process of Theorem 3.2, let $h \in \mathcal{K}(f)$. Then, for every $r \in P_w$, $1 \leq j \leq q$ and $\xi_{ea}(c_w^j(f)) < \gamma_w^j$ or $\xi_{ea}(c_w^j(f)) > \gamma_w^j$, we always have

$$(\xi_{ea}(c_w^j(f) - \gamma_w^j)(h_r^j - f_r^j) \geq 0.$$

Thus, the formula (21) holds and the proof is complete. □

Consequently, from Theorems 3.2 and 4.2, we have the following result.

Corollary 4.2 *Let $e \in int\mathcal{R}^l_+$. If there is $a \in \mathcal{R}^l$ such that a flow $f \in \mathcal{K}(f)$ is a solution of the following variational inequality:*

$$\sum_{j=1}^{q} \sum_{w \in W} \sum_{p \in P_w} \left\langle \xi_{ea}(c^j_p(f)), h^j_p - f^j_p \right\rangle \geq 0, \quad \forall h \in \mathcal{K}(f),$$

then the flow f is in weak vector equilibrium.

5 Conclusions

In this paper, we assumed that there are lower and upper capacity constraints for all paths and products. We introduced a (weak) vector equilibrium principle and obtained a sufficient condition for a (weak) vector equilibrium flow to be a solution for the system of (weak) vector quasi-variational inequalities. Since the necessary condition for a (weak) vector equilibrium flow to be a solution for the system of (weak) vector quasi-variational inequalities may not hold, we introduced a (weak) ξ-equilibrium flow by using Gerstewitz's nonconvex separation functional ξ. We proved a sufficient and necessary condition that a weak vector equilibrium flow is a weak ξ-equilibrium flow. We also established an equivalent relation between a weak vector equilibrium flow and a solution of a scalar variational inequality problem.

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